Choice rule $\Longleftrightarrow$ Preference relation

$C(\cdot)$ $\qquad$ $\succsim$

WARP $\qquad$ **Rational**

$x, y \in B$, $x \in C(B)$, $x \in B'$, $y \in C(B')$ $\Rightarrow x \in C(B')$

$\Downarrow$

$\succsim_c$

+) $\mathcal{B}$ includes all subsets of X with ≤ 3 elements

$x \succsim_c y \Leftrightarrow \exists B \in \mathcal{B}$ s.t. $x, y \in B$, $x \in C(B)$

Completeness: $\forall x, y \in X$, either $x \succsim y$ or $y \succsim x$

Transitivity: $\forall x, y, z \in X$, $x \succsim y$, $y \succsim z \Rightarrow x \succsim z$

$\Downarrow$

$C_\succsim$

$C_\succsim(B) = \{x \in B : x \succsim y \ \forall y \in B\}$

WARP is necessary but not sufficient for rationality.

1. Consider a choice problem with choice set $X = \{x, y, z\}$. Consider the following choice structures:

   (a) $(\mathscr{B}_1, C(\cdot))$, in which $\mathscr{B}_1 = \{\{x,y\},\{y,z\},\{x,z\},\{x\},\{y\},\{z\}\}$ and $C(\{x,y\}) = \{x\}, C(\{y,z\}) = \{y\}, C(\{x,z\}) = \{z\}, C(\{x\}) = \{x\}, C(\{y\}) = \{y\}, C(\{z\}) = \{z\}$.

   (b) $(\mathscr{B}_2, C(\cdot))$, in which $\mathscr{B}_2 = \{\{x,y,z\},\{x,y\},\{y,z\},\{x,z\},\{x\},\{y\},\{z\}\}$ and $C(\{x,y,z\}) = \{x\}, C(\{x,y\}) = \{x\}, C(\{y,z\}) = \{z\}, C(\{x,z\}) = \{z\}, C(\{x\}) = \{x\}, C(\{y\}) = \{y\}, C(\{z\}) = \{z\}$.

   (c) $(\mathscr{B}_3, C(\cdot))$, in which $\mathscr{B}_3 = \{\{x,y,z\},\{x,y\},\{y,z\},\{x,z\},\{x\},\{y\},\{z\}\}$ and $C(\{x,y,z\}) = \{x\}, C(\{x,y\}) = \{x\}, C(\{y,z\}) = \{y\}, C(\{x,z\}) = \{x\}, C(\{x\}) = \{x\}, C(\{y\}) = \{y\}, C(\{z\}) = \{z\}$.

   For every choice structure, comment on if the $WARP$ is satisfied and if there exists a rational preference relation $\succsim$ that rationalizes $C(\cdot)$ relative to its $\mathscr{B}$. If such a rationalization is possible, write it down.

(a) ① WARP is (trivially) satisfied.

the same couple never appears more than once in different budgets.

② $\mathscr{B}_1$ is not rationalizable.

$C(\{x,y\}) = \{x\}$ rationalized by $x \succsim_c y$

$C(\{y,z\}) = \{y\}$ 〃 〃 $y \succsim_c z$

$C(\{x,z\}) = \{z\}$ 〃 〃 $z \succsim_c x$

$P \to q$	
$P \wedge q$	T
$P \wedge \neg q$	F
$\neg P \wedge q$	T
$\neg P \wedge \neg q$	T

$\Rightarrow \succsim_c$ is not transitive (loop), thus not rational.

① & ② $\Rightarrow$ If $\mathscr{B}$ does not include all budgets up to 3 elements, $\succsim_c$ is not necessarily rational (even with WARP)

(b) ① WARP is not satisfied

$C(\{x,y,z\}) = \{x\}$

but $C(\{x,z\}) = \{z\}$

② $\mathscr{B}_2$ is not rationalizable

$C(\cdot)$ WARP + all ≤ 3 elements subsets $\Longleftrightarrow$ $\succsim_c$ rational

So $\succsim_c$ rational $\Rightarrow$ $C(\cdot)$ WARP

Contrapositive : $\neg$ WARP $\Rightarrow$ $\succsim_c$ not rational

(c) ① WARP is satisfied.

② $\mathcal{B}_3$ is rationalizable

$x \succsim_c y \succsim_c z$ is rational and rationalizes $(\mathcal{B}_3, C(\cdot))$.

Here $\mathcal{B}_3$ includes all budgets up to 3 elements

So $C(\cdot)$ WARP $\Rightarrow$ $\succsim_c$ rational

2. Define two alternative versions of $WARP$: assume $C(\cdot)$ on $(X, \mathscr{B})$ satisfies

$WARP^*$ iff for each $A, B \in \mathscr{B}, C(A) \cap B \neq \emptyset \Rightarrow C(B) \cap A \subset C(A)$

$WARP^{**}$ iff for each $A, B \in \mathscr{B}, x, y \in A \cap B, x \in C(A), y \notin C(A) \Rightarrow y \notin C(B)$

(a) Show that $WARP^*$ is equivalent to $WARP$.

(b) Show that $WARP^{**}$ is equivalent to $WARP$.

(Hint: Firstly verify that $WARP$ can be rewritten as: for each $A, B \in \mathscr{B}, x, y \in A \cap B, x \in C(A), y \in C(B) \Rightarrow y \in C(A)$ or $\Rightarrow x \in C(B)$.)

(a) pf: "$\Rightarrow$": Assume $x, y \in A \cap B$, $x \in C(A)$, $y \in C(B)$

(WTS: $y \in C(A)$)

$$\begin{cases} x \in C(A) \\ x \in B \end{cases} \Rightarrow x \in C(A) \cap B \Rightarrow C(A) \cap B \neq \phi$$

By $WARP^*$, $C(B) \cap A \subset C(A)$

$$\begin{cases} y \in C(B) \\ y \in A \end{cases} \Rightarrow y \in C(B) \cap A \subset C(A)$$

"$\Leftarrow$": Assume $C(A) \cap B \neq \phi$

Let $x \in C(A) \cap B$. take $\forall y \in C(B) \cap A$ (WTS: $y \in C(A)$)

$$\begin{cases} x \in C(A) \\ y \in C(B) \end{cases} \Rightarrow x, y \in A \cap B$$

By $WARP$, $y \in C(A) \Rightarrow C(B) \cap A \subset C(A)$

#

(b) pf: "$\Rightarrow$": Assume $x, y \in A \cap B$, $x \in C(A)$, $y \in C(B)$

(WTS: $y \in C(A)$)

Suppose $y \notin C(A)$, by WARP**, $\Rightarrow y \notin C(B)$ $\times$.

Thus, $y \in C(A)$

"$\Leftarrow$": Assume $x, y \in A \cap B$, $x \in C(A)$, $y \notin C(A)$

(WTS: $y \notin C(B)$)

Suppose $y \in C(B)$, by WARP, $y \in C(A)$ $\times$.

Thus, $y \notin C(B)$

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3. Show that if X is finite and $\succsim$ is rational, then $C_{\succsim}(B) \neq \emptyset$ for any $B \in \mathscr{B}$. (Hint: Use induction.)

pf: Suppose X has n elements. $n \in \mathbb{N}$.

① $n=1$. X has only one element, say x,

then $\mathscr{B} = \{\{x\}\}$, $B = \{x\}$.

By completeness of X, $x \succsim x$

Then $C_{\succsim}(B) = \{x\} \neq \phi$

② $n \geq 2$ Suppose the claim holds for n-element set.

Consider an $(n+1)$-element set X

Choose $x \in X$ and $X \setminus \{x\}$ has n element

Suppose $X \setminus \{x\}$ forms $\mathscr{B} = \{B_1, B_2, \cdots B_{2^n-1}\}$

$\Rightarrow C_{\succsim}(B_i) \neq \phi$ for $\forall B_i \in \mathscr{B}$

Consider $\forall B_i \in \mathscr{B}$. define $B_i' = B_i \cup \{x\}$

then X forms $\mathscr{B}' = \mathscr{B} \cup \{x\} \cup (\bigcup_{i=1}^{2^n-1} B_i')$

(WTS: $C_{\succsim}(B_i') \neq \phi$)

Consider B_i' for $\forall i \in \{1, 2, \cdots, 2^n-1\}$

Since $C_{\succsim}(B_i) \neq \phi$. suppose $y \in C_{\succsim}(B_i)$

By completeness of $\succsim$, either $x \succsim y$ or $y \succsim x$

Case I: $y \succsim x$. $y \in C_{\succsim}(B_i')$

Case II: $x \succsim y$, then as $y \succsim z$ for $\forall z \in B_i$

by transitivity, $x \succsim z$ for $\forall z \in B_i$

$\Rightarrow x \in C_{\succsim}(B_i)$

$\Rightarrow x \in C_{\succsim}(B_i \cup \{x\})$

$\Rightarrow x \in C_{\succsim}(B_i')$

In both cases, $C_{\succsim}(B_i') \neq \phi$

$\Rightarrow C_{\succsim}(B_k) \neq \phi$ for $\forall B_k \in \mathcal{B}'$

$k = \{1, 2, \cdots, 2^{n+1}-1\}$

By induction, if X is finite and $\succsim$ is rational,

$C_{\succsim}(B) \neq \phi$ for $\forall B \in \mathcal{B}$.

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4. Let X be a finite set with more than $N \geq 1$ elements, $\mathscr{B}$ its non-empty subsets, and $\succsim_1, \succsim_2$ two rational preference relations on X. Suppose that someone follows the following choice procedure: for each $B \in \mathscr{B}$, if B has more than N elements, then $C(B)$ is based on $\succsim_1$; if B has no more than N elements, then $C(B)$ is based on $\succsim_2$. Show that this choice rule violates $WARP$.

pf: Consider $X = \{x, y, z\}$

with $x \succsim_1 y \succsim_1 z$ and $y \succsim_2 x \succsim_2 z$

Set $N = 2$

$\Rightarrow C(\{x, y\}) = \{y\}$ $(\succsim_2)$

$C(\{x, y, z\}) = \{x\}$ $(\succsim_1)$

which violates $WARP$.

#

5. The *path-invariance* property has the following definition: For every pair $B_1, B_2 \in \mathscr{B}$ such that $B_1 \cup B_2 \in \mathscr{B}$ and $C(B_1) \cup C(B_2) \in \mathscr{B}$, we have $C(B_1 \cup B_2) = C(C(B_1) \cup C(B_2))$, that is, the decision problem can safely be subdivided.

(a) Show that a choice structure $(\mathscr{B}, C(\cdot))$ for which a rationalizing preference relation $\succsim$ exists satisfies the *path-invariance* property.

(b) Find examples of choice procedures that do not satisfy this property.

(a) Pf: (a) Let $\succsim$ rationalizes $C(\cdot)$.

"$\subseteq$": Choose $\forall x \in C_{\succsim}(B_1 \cup B_2)$

$\Rightarrow x \succsim y$ for $\forall y \in B_1 \cup B_2$ --- $(*)$

Since $x \in C_{\succsim}(B_1 \cup B_2)$, $x \in B_1 \cup B_2$

WLOG, assume $x \in B_1$

Clearly, $x \succsim a$ for $\forall a \in B_1$

$\Rightarrow x \in C_{\succsim}(B_1)$

$\Rightarrow x \in C_{\succsim}(B_1) \cup C_{\succsim}(B_2) \subseteq B_1 \cup B_2$

$\overset{\text{by } (*)}{\Rightarrow} x \succsim b$ for $\forall b \in C_{\succsim}(B_1) \cup C_{\succsim}(B_2)$

$\Rightarrow x \in C_{\succsim}(C_{\succsim}(B_1) \cup C_{\succsim}(B_2))$

$\Rightarrow C_{\succsim}(B_1 \cup B_2) \subseteq C_{\succsim}(C_{\succsim}(B_1) \cup C_{\succsim}(B_2))$

"$\supseteq$": Choose $\forall x \in C_{\succsim}(C_{\succsim}(B_1) \cup C_{\succsim}(B_2))$

$\Rightarrow x \succsim y$ for $\forall y \in C_{\succsim}(B_1) \cup C_{\succsim}(B_2)$ $\cdots (**)$

Since $x \in C_{\succsim}(C_{\succsim}(B_1) \cup C_{\succsim}(B_2))$, $x \in C_{\succsim}(B_1) \cup C_{\succsim}(B_2)$

WLOG, assume $x \in C_{\succsim}(B_1)$

$\Rightarrow x \succsim a$ for $\forall a \in B_1$ --- (1)

Moreover, consider $z \in C_{\succsim}(B_2) \subseteq C_{\succsim}(B_1) \cup C_{\succsim}(B_2)$

by (**) $\Rightarrow$ $x \succsim z$ for $\forall z \in C_{\succsim}(B_2)$

And $z \succsim b$ for $\forall b \in B_2$

By transitivity, $x \succsim b$ for $\forall b \in B_2$ -- (2)

Combining (1) & (2), $x \succsim c$ for $\forall c \in B_1 \cup B_2$

$\Rightarrow x \in C_{\succsim}(B_1 \cup B_2)$

$\Rightarrow C_{\succsim}\left(C_{\succsim}(B_1) \cup C_{\succsim}(B_2)\right) \subseteq C_{\succsim}(B_1 \cup B_2)$

Therefore, $C_{\succsim}(B_1 \cup B_2) = C_{\succsim}\left(C_{\succsim}(B_1) \cup C_{\succsim}(B_2)\right)$

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(b) Counter-example:

Consider the choice rule in Q4.

$C(\{x,y,z\}) = x \neq y = C\left(C(\{x,y\}) \cup C(\{z\})\right)$

6. Suppose that choice structure $(\mathscr{B}, C(\cdot))$ satisfies $WARP$. In the lecture, the *revealed (at-least-as-good-as) preference relation* $\succsim_C$ is defined by:

$$x \succsim_C y \Leftrightarrow \exists B \in \mathscr{B} \text{ s.t. } x, y \in B \text{ and } x \in C(B)$$

Consider the following other two possible revealed preferred relations, $\succ^*$ and $\succ^{**}$:

$$x \succ^* y \Leftrightarrow \exists B \in \mathscr{B} \text{ s.t. } x, y \in B, x \in C(B), \text{ and } y \notin C(B)$$

$$x \succ^{**} y \Leftrightarrow x \succsim_C y \text{ but not } y \succsim_C x$$

(a) Show that $\succ^*$ and $\succ^{**}$ give the same relation over X; that is, for any $x, y \in X$, $x \succ^* y \Leftrightarrow x \succ^{**} y$. Is this still true if $(\mathscr{B}, C(\cdot))$ does not satisfy $WARP$?

(b) Must $\succ^*$ be transitive?

(c) Show that if $\mathscr{B}$ includes all subsets of X up to 3 elements, then $\succ^*$ is transitive.

(a) pf: ① "$\Rightarrow$": Assume $x \succ^* y$

$\Rightarrow \exists B \in \mathscr{B}$ s.t. $x, y \in B$, $x \in C(B)$, $y \notin C(B)$

By definition of $\succsim_C$, $x \succsim_C y$

Suppose $y \succsim_C x$.

Consider B, since $x, y \in B$, $x \in C(B)$,

by WARP, $y \in C(B)$ $\times$!

Thus, $x \succsim_C y$ but not $y \succsim_C x$

$\Rightarrow x \succ^{**} y$

"$\Leftarrow$": Assume $x \succ^{**} y$.

then $x \succsim_C y$ but not $y \succsim_C x$

$\Rightarrow \exists B \in \mathscr{B}$ s.t. $x, y \in B$, $x \in C(B)$

and $\neg(\exists B' \in \mathscr{B}$ s.t. $x, y \in B'$, $y \in C(B'))$

$\Leftrightarrow \forall B' \in \mathscr{B}$ s.t. $x, y \in B'$, $y \notin C(B')$

$\Rightarrow \exists B \in \mathcal{B}$ s.t. $x, y \in B$, $x \in C(B)$, $y \notin C(B)$

By definition, $x \succ^* y$ #

② No.

From ①, WARP is not necessary for "$\Leftarrow$",

but is necessary for "$\Rightarrow$".

Counter-example:

$X = \{x, y, z\}$, $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$

$C(\{x, y\}) = \{y\}$, $C(\{x, y, z\}) = \{x\}$

Then $x \succ^* y$ and $y \succ^* x$.

but neither $x \succ^{**} y$ nor $y \succ^{**} x$

since $x \succsim_c y$ and $y \succsim_c x$

(b) No.

Counter-example:

$X = \{x, y, z\}$. $\mathcal{B} = \{\{x, y\}, \{y, z\}\}$

$C(\{x, y\}) = \{x\}$. $C(\{y, z\}) = \{y\}$

Then $x \succ^* y$, $y \succ^* z$

but we do not have $x >^* z$

since $\nexists\ B \in \mathcal{B}$ s.t. $x, z \in B$, $x \in C(B)$, $z \notin C(B)$

(c) pf: Since $\mathcal{B}$ includes all subsets up to 3 elements

and by WARP, $\succsim_c$ is rational, thus transitive.

$\Rightarrow\ >^{**}$ is also transitive

(See Q8 (a) (2) for a similar proof)

By (a), $>^*$ is equivalent to $>^{**}$

$\Rightarrow\ >^*$ is transitive

#

7. Monotonicity and nonsatiation are two properties of $\succsim$. This exercise investigates the relationship between them. Suppose $\succsim$ is defined on the consumption set $X = R_+^L$. According to MWG, we have the following definitions regarding monotonicity in a slightly different way from the definitions in the lecture:

$\succsim$ on X is monotone if $x, y \in X, y >> x \Rightarrow y \succ x$

$\succsim$ on X is strongly monotone if $x, y \in X, y \geq x$ and $y \neq x \Rightarrow y \succ x$

$\succsim$ on X is weakly monotone if $x, y \in X, y \geq x \Rightarrow y \succsim x$

where $y >> x$ means that every element of y is greater than every element of x.

(a) Show that if $\succsim$ is strongly monotone, then it is monotone.

(b) Show that if $\succsim$ is monotone, then it is locally nonsatiated. (Hint: For $x, y \in R_+^L$, the Euclidean distance between x and y is defined as $||x-y|| = [\sum_{l=1}^{L}(x_l - y_l)^2]^{\frac{1}{2}}$.)

(c) Draw a convex preference relation that is locally nonsatiated but is not monotone to show that the converse proposition of (b) does not hold, that is, local nonsatiation is a weaker assumption than monotonicity.

(d) Show that if $\succsim$ is transitive, locally nonsatiated, and weakly monotone, then it is monotone.

(a) pf: Assume $x, y \in X$, $y >> x$, then $y \geq x$ and $y \neq x$

Since $\succsim$ is strongly monotone, $y \succ x$

#

(b) Def of LNS: $\forall x \in X$. $\forall \varepsilon > 0$, $\exists y \in X$ s.t. $y \succ x$, $||y - x|| < \varepsilon$

pf: Consider $x \in \mathbb{R}^L$ and $\forall \varepsilon > 0$,

take $y = x + \frac{\varepsilon}{\sqrt{L+1}} e \in \mathbb{R}^L$ where $e = (1, \cdots, 1) \in \mathbb{R}^L$

then. ① Since $y >> x$. by monotonicity of $\succsim$, $y \succ x$

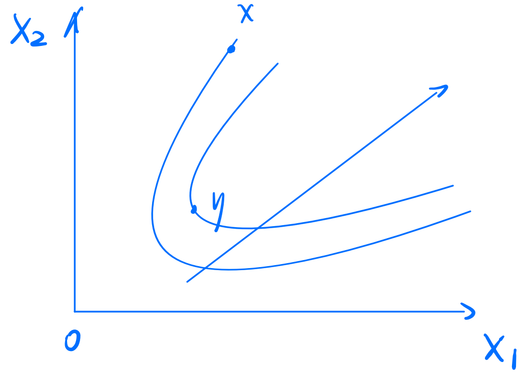
② $||y - x|| = \sqrt{\sum_{l=1}^{L}(y_l - x_l)^2} = \sqrt{L \cdot (\frac{\varepsilon}{\sqrt{L+1}})^2}$

$= \sqrt{\frac{L}{L+1}\varepsilon^2} < \sqrt{\varepsilon^2} = \varepsilon$

By definition. $\succsim$ is LNS.

#

(c)



$X = \mathbb{R}_+^2$

$x \gg y$

but $y \succ x$

(d) pf: Assume $x, y \in X$, $x \gg y$.

Consider $\varepsilon = \min\{x_1 - y_1, \cdots, x_L - y_L\} > 0$

then for $\forall z \in X$, if $\|y - z\| < \varepsilon$, $x \gg z$

By LNS, $\forall \varepsilon > 0$, $\exists z' \in X$ s.t. $z' \succ y$, $\|y - z'\| < \varepsilon$

$\Rightarrow x \gg z'$

$\Rightarrow x \geq z'$

By weakly monotonicity of $\succsim$, $x \succsim z'$

$\Rightarrow x \succsim z' \succ y$

By transitivity (HW1 Q1), $x \succ y$

$\Rightarrow \succsim$ is monotone

#

8. Let $\succsim$ be a preference relation on a set X. Suppose $\succsim$ is complete and transitive. Recall the definitions of $\succ$ and $\sim$ derived from $\succsim$ in the lecture.

(a) Show that $\succ$ is:

$x \succ y$ iff $x \succsim y$ and $y \not\succsim x$

(1) Irreflexive: $x \succ x$ never holds

(2) Transitive: $x \succ y, y \succ z \Rightarrow x \succ z$

(3) Asymmetric: $x \succ y \Rightarrow y \not\succ x$

(4) Satisfying negative transitivity: $x \not\succ y, y \not\succ z \Rightarrow x \not\succ z$

(b) Show that $\sim$ is:

$x \sim y$ iff $x \succsim y$ and $y \succsim x$

(1) Reflexive: $x \sim x$ always holds

(2) Transitive: $x \sim y, y \sim z \Rightarrow x \sim z$

(3) Symmetric: $x \sim y \Rightarrow y \sim x$

(c) Define $I(x)$ to be the set of all $y \in X$ for which $y \sim x$. Show that the set $\{I(x) | x \in X\}$ is a partition of X, that is,

(1) $\forall x \in X, I(x) \neq \emptyset$

(2) $\forall x \in X, \exists y \in X$ such that $x \in I(y)$

(3) $\forall x, y \in X,$ either $I(x) = I(y)$ or $I(x) \cap I(y) = \emptyset$

(a) (1) pf: Suppose $x \succ x$ holds.

then $x \succsim x$ and $x \not\succsim x$ ※.

Thus, $x \succ x$ never holds

#

(2) pf: Assume $x \succ y$. $y \succ z$

$\Rightarrow$ ① $x \succsim y$, ② $y \not\succsim x$, ③ $y \succsim z$, ④ $z \not\succsim y$

By transitivity. ① & ③ $\Rightarrow$ $x \succsim z$

Suppose $z \succsim x$. by ① $\Rightarrow$ $z \succsim y$

contradicting ④ ※.

Thus, $z \not\succsim x$

$\Rightarrow \begin{cases} x \succsim z \\ z \not\succsim x \end{cases} \Rightarrow x \succ z$

#

(3) Pf: Assume $x \succ y$, then $x \succsim y$, $y \not\succsim x$

Suppose $y \succ x$, then $y \succsim x$, $x \not\succsim y$ ※.

$\Rightarrow y \not\succ x$

#

(4) Pf: First show that $x \not\succ y \Longleftrightarrow y \succsim x$

"$\Rightarrow$": $x \not\succ y \Rightarrow x \not\succsim y$ or $y \succsim x$

By completeness of $\succsim$, either $x \succsim y$ or $y \succsim x$

$\Rightarrow y \succsim x$

"$\Leftarrow$": Assume $y \succsim x$.

Suppose not, i.e., $x \succ y$

$\Rightarrow x \succsim y$ and $y \not\succsim x$ ※.

$\Rightarrow x \not\succ y$

Thus, for $\forall x, y$, $x \not\succ y \Longleftrightarrow y \succsim x$

Assume $x \not\succ y$, $y \not\succ z$

$\Longleftrightarrow y \succsim x$, $z \succsim y$

By transitivity of $\succsim$, $z \succsim x$

$\Longleftrightarrow x \not\succ z$ #

(b) (1) pf: Take $y = x$, by completeness of $\succsim$.

either $x \succsim x$ or $x \precsim x$

$\Rightarrow x \succsim x$

$\Rightarrow x \succsim x$ and $x \precsim x$

$\Rightarrow x \sim x$

#

(2) pf: Assume $x \sim y$, $y \sim z$,

$\Rightarrow x \succsim y$, $y \succsim x$, $y \succsim z$, $z \succsim y$

By transitivity, $\left.\begin{array}{l} \begin{cases} x \succsim y \\ y \succsim z \end{cases} \Rightarrow x \succsim z \\ \begin{cases} y \succsim x \\ z \succsim y \end{cases} \Rightarrow z \succsim x \end{array}\right\} \Rightarrow x \sim z$

#

(3) pf: Assume $x \sim y$,

$\Rightarrow x \succsim y$, $y \succsim x$

$\Rightarrow y \succsim x$, $x \succsim y$

$\Rightarrow y \sim x$

#

(c) (1)

pf: Choose $\forall x \in X$, by reflexitivity of "$\sim$" in (b) (1)

$$x \sim x \Rightarrow x \in I(x)$$

$$\Rightarrow I(x) \neq \phi$$

#

(2) pf: Choose $\forall x \in X$, by (1), $x \in I(x)$

Take $y = x \in X \Rightarrow x \in I(y) = I(x)$

#

(3) pf: Choose $\forall x, y \in X$.

Assume $I(x) \cap I(y) \neq \phi$ i.e., $\exists z \in I(x) \cap I(y)$

WTS: $I(x) = I(y)$

"$\subseteq$": Choose $\forall a \in I(x)$ (WTS: $a \in I(y)$)

$\Rightarrow a \sim x$

Also, since $z \in I(x) \cap I(y)$. $z \in I(x) \Rightarrow z \sim x$

By transitivity of $\sim$ in (b) (2), $a \sim z$

In addition, $z \in I(y) \Rightarrow z \sim y$

$$\left.\begin{array}{l} a \sim z \\ z \sim y \end{array}\right\} \Rightarrow a \sim y \Rightarrow a \in I(y)$$

Thus, for $\forall a \in I(x)$, we have $a \in I(y)$

$$I(x) \subseteq I(y)$$

"$\supseteq$": can be proved in a similar way.

(Choose $\forall b \in I(y)$, $\Rightarrow b \in I(x)$)

Consequently, if $I(x) \cap I(y) \neq \phi$, $I(x) = I(y)$

#