Today's discussion is on logic as covered in this week's lectures. First, directions for discussion:

- Discussion includes several exercises based on recent course material to be done **without** any electronic devices. You will work with at least one person sitting near you. Take a few minutes to get to know each other.
- You are expected to actively participate in all exercises to receive credit for discussion.
- If you have completely finished all of the exercises before the share stage, please show this to the instructor and ask permission before opening an electronic device to do other work.
- The order of operations and a list of axiomatic rules is at the end of this handout and is available separately on Canvas as `equiv.pdf`.

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1. Recall De Morgan's laws: $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$ and $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$. As we discussed in lecture, English grammar does not line up perfectly. In English, we have common way of using the former. If we say "It is not this or that," formally speaking we are actually saying "it is not this, or it **is** that" due to precedence of operators. However, "It is neither this nor that," it is understood to mean "it is not this and it is not that." That is, the logical statement "not either (A or B)" is the same as the English phrase "neither A nor B", which is the same as "(not A) and (not B)."

   (a) Argue that the following English statements are equivalent by introducing propositional variables for parts of each statement and using De Morgan's laws.

      i. "It is neither raining outside, nor is it sunny outside." and "It is not raining outside and it is not sunny outside."

      ii. "A positive integer x cannot be both even and odd." and "A positive integer x is either not even, or x is not odd."

   (b) Write down an equivalent English statement for each of the following:

      i. "A cat in a closed box is not both alive and dead."

      ii. "The season cannot be both winter and summer."

      iii. "The swimmer is neither running nor jumping."

2. Prove that $P \Rightarrow Q$ is indeed equivalent to its *contrapositive*, $\neg Q \Rightarrow \neg P$ by (i) comparing the statements' truth tables and (ii) by a chain of equivalence rules. For the latter, please annotate your steps with the rule being applied.

3. Prove that $P \vee Q \Rightarrow R$ is equivalent to $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ by (i) comparing the statements' truth tables and (ii) by a chain of equivalence rules. For the latter, please annotate your steps with the rule being applied.

4. Goldbach's Conjecture is "Every even integer greater than 2 can be written as the sum of two primes." Indeed, this claim remains unsolved today!

   - Write Goldbach's Conjecture as a quantified predicate formula in terms of predicates that you define, such as $A(x)$ = "x is divisible by 230". Be sure to specify the domain of your variables. **For purposes of this exercise, ensure that all quantifiers come at the front of your formula (as opposed to nested within).** Recall that $\mathbb{Z}$ is the set of all integers and $\mathbb{R}$ is the set of all reals.
   - Consider taking your formula, but reversing the order of your quantifiers. What does it mean in English? Can you determine whether it is True or False?

Logical Equivalences

Order of operations by precedence: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$.

$A \wedge B \Leftrightarrow B \wedge A$	(commutativity of $\wedge$)
$A \vee B \Leftrightarrow B \vee A$	(commutativity of $\vee$)
$(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$	(associativity of $\wedge$)
$(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$	(associativity of $\vee$)
$A \wedge A \Leftrightarrow A$	(idempotence for $\wedge$)
$A \vee A \Leftrightarrow A$	(idempotence for $\vee$)
$A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$	(distributivity of $\wedge$ over $\vee$)
$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$	(distributivity of $\vee$ over $\wedge$)
$\neg\neg A \Leftrightarrow A$	(double negation)
$A \Rightarrow B \Leftrightarrow \neg A \vee B$	(implication in terms of $\vee$)
$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$	(De Morgan for $\wedge$)
$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$	(De Morgan for $\vee$)
$A \Rightarrow B \Leftrightarrow \neg B \Rightarrow \neg A$	(contrapositive)
$A \vee \neg A \Leftrightarrow \mathbf{T}$	(tautology)
$A \wedge \neg A \Leftrightarrow \mathbf{F}$	(contradiction)

Reminders:

- Everyone should have access to Canvas, Ed, and Gradescope. Please contact Alex TODAY if you do not have access, and in particular, if you are trying to enroll but need a permission number.
- Lecture notes are typically available by end-of-day.
- HW1 is released on Canvas and is due next Wednesday 1/22 at 5pm.
- Office hours start tomorrow (Saturday), with slots on Sat/Sun (virtual) and Mon-Wed (in-person) each week. The schedule is on the course website.

Recap:

* Proposition (no unknown circumstances) vs predicate ?
* Logical Operators

  $\neg$, $\vee$, $\wedge$ "3 basics"

  $\neg A = \bar{A}$, $A \oplus B$ xor

  $\Rightarrow$: "If..., then...", $A \Rightarrow B \Leftrightarrow \neg A \vee B$

  $\Leftrightarrow$: "..iff..".

  Construct new operators (HW).

* Axiomatic rules (distributivity, De Morgan, --)
* Logical equivalence ① Truth table ② Chain of equivalence

  Truth table $\longleftrightarrow$ logical formula

  (easy to see, hard to write) (easy to write, hard to see)

1. Recall De Morgan's laws: $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$ and $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$. As we discussed in lecture, English grammar does not line up perfectly. In English, we have common way of using the former. If we say "It is not this or that," formally speaking we are actually saying "it is not this, or it **is** that" due to precedence of operators. However, "It is neither this nor that," it is understood to mean "it is not this and it is not that." That is, the logical statement "not either (A or B)" is the same as the English phrase "neither A nor B", which is the same as "(not A) and (not B)."

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(a) i. neither A nor B $\overset{?}{\Longleftrightarrow}$ ✓ not A and not B

$\Updownarrow$ (neither A nor B) $\Updownarrow$ (not A and not B)

not either A or B $\qquad\qquad$ $\neg A \wedge \neg B$

$\Updownarrow$

$\neg(A \vee B)$ $\overset{\text{De Morgan}}{\Longleftrightarrow}$ $\neg A \wedge \neg B$

ii. not both A and B $\overset{?}{\Longleftrightarrow}$ ✓ either not A or not B

$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

$\neg(A \wedge B)$ $\underset{\text{De Morgan}}{\Longleftrightarrow}$ $\neg A \vee \neg B$

   (b) Write down an equivalent English statement for each of the following:

      i. "A cat in a closed box is not both alive and dead."

      ii. "The season cannot be both winter and summer."

      iii. "The swimmer is neither running nor jumping."

i. ... is either not alive or not dead.

ii. ... is either not winter or not summer.

iii. ... is not running and is not jumping.

2. Prove that $P \Rightarrow Q$ is indeed equivalent to its *contrapositive*, $\neg Q \Rightarrow \neg P$ by (i) comparing the statements' truth tables and (ii) by a chain of equivalence rules. For the latter, please annotate your steps with the rule being applied.

"Proof by contrapositive"

e.g. Prove: If n is even, then n^2 is even.

$n = 2k$, $n^2 = 4k^2$, $k \in \mathbb{Z}$

If n^2 is even, then n is even

$n^2 = 2k$, $n = \sqrt{2k}$?

$\Longleftrightarrow$ If n is odd, then n^2 is odd.

$n = 2k+1$. $n^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$ ✓

Pf: (i)

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

(ii) $P \Rightarrow Q$

$\Longleftrightarrow \neg P \vee Q$ implication

$\Longleftrightarrow Q \vee \neg P$ commutativity of $\vee$

$\Longleftrightarrow \neg(\neg Q) \vee (\neg P)$ double negation

$\Longleftrightarrow \neg Q \Rightarrow \neg P$ implication

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$2^3 = 8$

3. Prove that $P \vee Q \Rightarrow R$ is equivalent to $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ by (i) comparing the statements' truth tables and (ii) by a chain of equivalence rules. For the latter, please annotate your steps with the rule being applied.

Pf: (i)

P	Q	R	$P \vee Q$	$P \vee Q \Rightarrow R$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	T	T	T	T	T	T
T	T	F	T	F	F	F	F
F	F	T	F	T	T	T	T
F	T	F	T	F	T	F	F
T	F	F	T	F	F	T	F
F	F	F	F	T	T	T	T

(ii) $P \vee Q \Rightarrow R$

$\Leftrightarrow \neg(P \vee Q) \vee R$ implication

$\Leftrightarrow (\neg P \wedge \neg Q) \vee R$ De Morgan's law

$\Leftrightarrow (\neg P \vee R) \wedge (\neg Q \vee R)$ distributivity of $\vee$ over $\wedge$

$\Leftrightarrow (P \Rightarrow R) \wedge (Q \Rightarrow R)$ implication

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* Intro to predicate logic.

- Def. Predicates are statements with variables, whose truth value depends on these variables.

  Ex. ① $P(x) =$ "x is odd." ; $Q(x) =$ "x is even"

  $P(1)$, $Q(2)$ are true.

  $P(2)$, $Q(1)$ are false.

  ② $P(x,y) =$ "$x+2=y$."

  $P(1,3)$ is true.

  $P(1,4)$ is false.

Predicates $\neq$ propositions.

$\Rightarrow$ specify the input

$\Rightarrow$ domain.

- Def. The domain of a variable is the set of all values that may be substituted in place of the variable.

  Ex. ① $x \in \mathbb{Z}$

  ② $x \in \mathbb{R}$, $y \in \mathbb{R}$

- Two quantifiers (interact with predicates & domains)

  $\Rightarrow$ { The universal quantifier, $\forall$ "for all"
  The existential quantifier, $\exists$ "There exists"

  Ex. ① $\forall x \in \mathbb{Z},\ P(x) \vee Q(x)$ $\Rightarrow$ True.

  $\Leftrightarrow$ For all integers x, x is even or x is odd.

  negation

  ② $\exists x \in \mathbb{Z},\ P(x) \wedge Q(x)$ $\Rightarrow$ False

  $\neg \forall x\ P(x) \equiv \exists x\ \neg P(x)$

$\Leftrightarrow$ There exists an integer x such that x is even and x is odd.

Remark 1: predicate + quantifier for each variable

$\Rightarrow$ variables "bound" or "defined"

$\Rightarrow$ proposition

Remark 2: The truth of a predicate depends on the domain.

Ex. ① $\forall x \in \mathbb{Z}, \; x^2 \geq x$ True.

$\forall x \in \mathbb{R}, \; x^2 \geq x$ False. $x = \frac{1}{2}$

② $\exists y \in \mathbb{R}, \; y^2 = y$ True. $y = 0$ or $y = 1$

$\exists y \in [2,3], \; y^2 = y$ False.

Remark 3: A predicate formula can have multiple quantifiers.

Ex. $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \; xy \leq y^2$ True. $x = 0 \Rightarrow 0 \leq y^2$

$\Leftrightarrow$ There exists an integer x such that for all integer y, xy is at most y^2.

- How to translate Math Theorems into a logical formula?

Ex. Fermat (1637): If an integer is greater than 2, then the equation $a^n + b^n = c^n$ has no solutions in positive integers a, b, and c.

Step 1: Identify 3 IMPORTANT elements:

① Variables: a, b, c, n

② Domains: $a, b, c \in \mathbb{Z}^+$, $n \in \mathbb{Z}$, $n > 2$

③ Conditions: $a^n + b^n \neq c^n$

Step 2: Define predicates

$A(x) =$ "$x \in \mathbb{Z}$ and $x > 2$".

$S(a, b, c, n) =$ "$a^n + b^n \neq c^n$".

Step 3: Add quantifiers & reconcile the order.

$\forall a, b, c \in \mathbb{Z}^+,\ A(n) \Rightarrow S(a, b, c, n)$

Remark: many ways to express

e.g., $\forall a, b, c \in \mathbb{Z}^+.\ \neg\big((n \in \mathbb{Z}) \wedge (n > 2)\big) \vee (a^n + b^n \neq c^n)$

$\Updownarrow$

$(n \in \mathbb{Z}) \vee (n \leq 2) \vee (a^n + b^n \neq c^n)$

4. Goldbach's Conjecture is "Every even integer greater than 2 can be written as the sum of two primes." Indeed, this claim remains unsolved today!

x

y, z

- Write Goldbach's Conjecture as a quantified predicate formula in terms of predicates that you define, such as $A(x)$ = "x is divisible by 230". Be sure to specify the domain of your variables. **For purposes of this exercise, ensure that all quantifiers come at the front of your formula (as opposed to nested within).** Recall that $\mathbb{Z}$ is the set of all integers and $\mathbb{R}$ is the set of all reals.
- Consider taking your formula, but reversing the order of your quantifiers. What does it mean in English? Can you determine whether it is True or False?

(a) Step 1: ① Variables : x, y, z

② Domains : $x \in \mathbb{Z}$, x even, y, z prime

③ Conditions: $x = y + z$

Step 2: $A(x)$ = "x is even and $x > 2$".

$P(x)$ = "x is prime".

$S(x, y, z)$ = "$x = y + z$".

Step 3:

$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \exists z \in \mathbb{Z}, A(x) \Rightarrow \big(P(y) \wedge P(z) \wedge S(x, y, z)\big)$

(b) $\exists z \in \mathbb{Z}, \exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}, A(x) \Rightarrow \big(P(y) \wedge P(z) \wedge S(x, y, z)\big)$

$\Rightarrow$ There exist prime numbers such that their sum is equal to any even integer greater than 2.

$\Rightarrow$ obviously false.

Order of quantifiers matters!

Extension: Writing logical formula for a truth table

Truth tables $\overset{\text{easy}}{\longleftarrow}\!\!\overset{\text{hard}}{\longrightarrow}$ Logical formula

$\Rightarrow$ New definitions of logical operators (HW).

	A	B	$A \Rightarrow B$
$A \wedge B$	T	T	T
$A \wedge \neg B$	T	F	F
$\neg A \wedge B$	F	T	T
$\neg A \wedge \neg B$	F	F	T

$\longleftrightarrow$

A	B	$\neg A$	$\neg A \vee B$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

① Look at the true row: take "or"

② Look at the false row: negate, take "and".

e.g.: ②: $\neg(A \wedge \neg B) \Leftrightarrow \neg A \vee (\neg\neg B) \Leftrightarrow \neg A \vee B$

$A \oplus B \equiv (A \vee B) \wedge \neg(A \wedge B)$

A	B	$A \oplus B$	$A \vee B$	$\neg(A \wedge B)$	
T	T	F	T	F	F
T	F	T	T	T	T
F	T	T	T	T	T
F	F	F	F	T	F

① $(A \wedge \neg B) \vee (\neg A \wedge B)$

② $\neg(A \wedge B) \wedge \underline{\neg(\neg A \wedge \neg B)}$

↳ $A \vee B$